Decoupling process for better synchronizability on scale-free networks

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We propose a decoupling process performed in scale-free networks to enhance the synchronizability of the network, together with preserving the scale-free structure. Simulation results show that the decoupling process can effectively promote the network synchronizability, which is measured in terms of eigenratio of the coupling matrix. Moreover, we investigate the correlation between some important structural properties and the collective synchronization, and find that the maximum vertex betweenness seems to be the most strongly correlated with the synchronizability among the major structural features considered. We explain the effect of the decoupling process from a viewpoint of coupling information transmission. Our work provides some evidence that the dynamics of synchronization is related to that of information or vehicle traffic. Because of the low cost in modifying the coupling network, the decoupling process may have potential applications.

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Complex networks can describe a wide range of systems from nature to society. Prototypical examples cover as diverse as the Internet, scientific collaboration networks, neural networks and gene regulation networks [1–6]. A large amount of empirical observations indicate that many networks share two common structural features, i.e., small-world (SW) [4] and scale-free (SF) [1] properties. Understanding the effect of networks on the dynamical processes taking place on them is a central issue [3]. Intensive investigations reveal that small-world and scale-free properties play significant roles in variety of dynamical processes [6].

Synchronization behavior is ubiquitous in natural and social systems. Much attention has been given to the oscillator networks [7], which are natural representations of real systems consisting of coupling units. Previously reported results have shown that the ability to synchronize is remarkably promoted in both SW and SF networks compared to regular networks, which is ascribed to the short average distance in SW and SF networks [8]. While in contrast to SW networks, SF networks tends to inhibit the synchronization, even though the average distance in SF networks is smaller than in SW networks [9]. This interesting result indicates that heterogeneity of degree distribution suppresses the synchronization, and the degree distribution may be more significant than the average distance for better synchronizability. More recently, oscillator networks with weighted coupling strength have drawn much interest. The influence of coupling strength on the collective synchronization has been investigated, and different coupling strength assignments have been proposed to enhance the synchronizability [10]. The advantage of weighted coupling is that the system can achieve the maximum synchronizability with keeping scale-free networks fixed, so that the advantages of scale-free structure in other dynamics can be held, such as navigation.

In this paper, we concentrate on the synchronizability of scale-free networks. Aiming to enhance the synchronizability, we propose a decoupling process. Simulation results show that by performing this process, synchronizability of scale-free networks is considerably promoted, together with unchanged degree distribution and network size. Our work also reveals that the synchronizability of scale-free networks is closely related to some key edges. In order to understand the effect of the decoupling process on synchronizability, we study the changes of some important structural characteristics induced by the process, including average distance, clustering coefficient, assortative mixing, and the maximum vertex betweenness. Simulation results show that the maximum vertex betweenness seems to be a good predictor for synchronizability.

We first introduce a generic model of oscillators placed on networks with edges representing couplings. We consider a diffusively coupled dynamical network consisting of identical oscillators. The state of ith oscillators is controlled by the following equation:

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma(x_j), \quad i = 1,2, \ldots ,N, \tag{1}$$

where $f(x)$ describes the dynamics of each individual oscillator, $\Gamma(x)$ is the output function, $A=(a_{ij})$ is the coupling matrix, and c is the coupling strength.

In the case of symmetric and unweighted coupling networks, the elements of A is defined as (i) if vertex i and j are connected, then $a_{ij}=a_{ji}=1$; (ii) otherwise, $a_{ij}=a_{ji}=0$. The diagonal entries are $a_{ii}=-k_i$, where k_i is degree of vertex i. In graph theory, $-A$ is usually called Laplace matrix. In the case of connected networks, A is negative semidefinite, so that all the eigenvalues of A are nonpositive real values and the largest eigenvalue is zero. The synchronization manifold is an invariant manifold, i.e., the completely synchronized state $x_1(t)=x_2(t)=\cdots=x_N(t)=s(t)$ satisfies $\dot{s}(t)=f(s(t))$,

Linearize Eq. (1) about $s(t)$ [12], we get

$$\dot{\delta} = Df(s)\delta_i + \sum_{j=1}^{N} ca_{ij}D\Gamma(s)\delta_i, \quad i = 1,2, \ldots ,N, \tag{2}$$

where $Df(s)$ and $D\Gamma(s)$ are the Jacobi matrixes of $f(s)$ and $\Gamma(s)$ about s, respectively, δ_i is the variation of x_i. Let $\Delta=[\delta_1,\delta_2,\ldots,\delta_N]$, Eq. (2) is rewritten as

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$$\dot{\Delta} = Df(s)\Delta + cD\Gamma(s)\Delta A^T, \quad (3)$$

where $A^T = S\Lambda S^{-1}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, and $\{\lambda_k\}_{k=1}^N$ is the eigenvalues of matrix A and $\lambda_1 = 0$. Then, let $\Delta S = [\eta_1, \eta_2, \ldots, \eta_N]$, and we get

$$\dot{\eta}_k = [Df(s) + c\lambda_k D\Gamma(s)]\eta_k, \quad k = 2,3,\ldots,N. \quad (4)$$

Note that in Eq. (4), only λ_k and η_k depend on k. In the case of symmetric coupling, i.e., $a_{ij} = a_{ji}$, the main stability equation of the system can be defined as

$$\dot{y} = [Df(s) + c\alpha D\Gamma(s)]y. \quad (5)$$

The largest Lyapunov exponent $L_{\max}$ of Eq. (5) is a function of α, and $L_{\max}$ is named main stability function, which determines the linear stability of the synchronized state. The synchronized state is stable if $L_{\max} < 0$ [12]. The eigenvalue λ_1 corresponds to a mode parallel to the synchronization manifold.

For many oscillatory dynamical systems, $L_{\max}$ is negative in a single, finite interval (α_1, α_2). The network is thus synchronizable for some c if the condition $\alpha_1 < c\lambda_k < \alpha_2$ is satisfied so that $L_{\max}(c\lambda_k) < 0$ for all $k \geq 2$. This is equivalent to the condition

$$R \equiv \lambda_N/\lambda_2 < \alpha_2/\alpha_1, \quad (6)$$

where eigenratio R depends only on the network structure and the range from α_1 to α_2 depends on the dynamics. Then synchronizability can be investigated through the simple eigenratio of coupling matrix. The smaller the eigenratio of the matrix, the stronger the synchronizability of the network [12].

Here, we introduce the decoupling process (DP) performed in scale-free networks. Without losing generality, we construct the scale-free network by using the simplest and well-known Barabási-Albert (BA) network model [13]. In this model, starting from m_0 fully connected vertices, one vertex with m edges is attached at each time step in such a way that the probability Π_i of being connected to the existing vertex i is proportional to the degree k_i of vertex i, i.e., $\Pi_i = k_i/\Sigma_j k_j$, where the sum runs over all the existing vertices. The typical structural characteristic of BA networks is that the degree distribution follows a power law $P(k) \sim k^{-\gamma}$ with $\gamma = 3$, which is independent of parameter m. m only controls the connectivity density of BA networks [1]. The DP is realized among small number of key edges. We define the significance of an edge G_{ij} by the product of the degrees of two vertices i and j at both sides of the edge, i.e., $G_{ij} = k_i k_j$. After calculating the significance of all the edges, we rank the edges according to their values G_{ij}. Subsequently, at each time step, we cut an edge with the highest rank, i.e., decouple the two vertices at both sides of the edge. Thus, we can calculate the synchronizability of the network in terms of the eigenratio of the coupling matrix and observe the change of structural features induced by the DP. Then repeating this process, the correlation between the synchronizability and the number of cut edges N_{cut}, as well as structural features as a function of N_{cut} can be obtained.

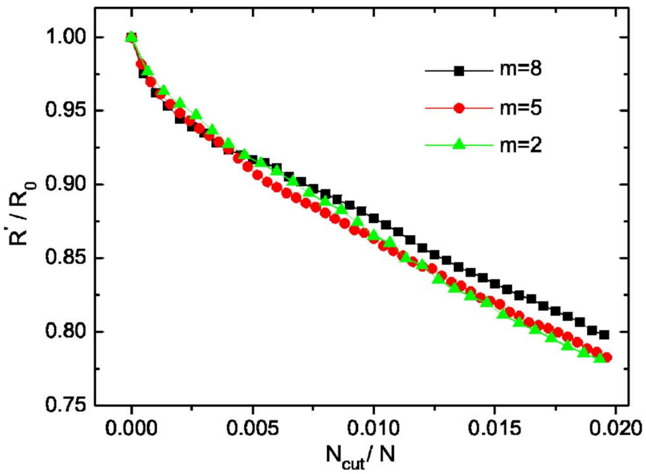

FIG. 1. (Color online) The change of synchronizability as a function of the proportion of cut edges N_{cut}/N for different values of m. Network size $n = 2000$.

All simulations are carried out for the network size $N = 2000$. Each data point is obtained by averaging over 20 network realizations. Figure 1 shows the ratio of the synchronizability R'/R_0 after and before the DP as a function of the proportion of cut edges N_{cut}/N. One can see that as N_{cut} increases, the synchronizability is considerably promoted, which is reflected by the decrease of the eigenratio. Moreover, for different values of m, R'/R_0 vs N_{cut}/N exhibits nearly the same decreasing trend with slight difference. The lower the value of m, the slightly faster the decreasing velocity. This m-independent behavior provides a criterion for the correlation between the synchronizability and the structural characteristics. Suppose that the change of a structural feature for different m demonstrates remarkable distinctions; it means that such structural feature is independent of the network synchronizability.

We turn to the effect of the DP on the network structure. We first investigate the degree distribution of the networks before and after the DP. As shown in Fig. 2, the degree distribution remains unchanged, so that the DP has slight influence on the scale-free property of the network. Average

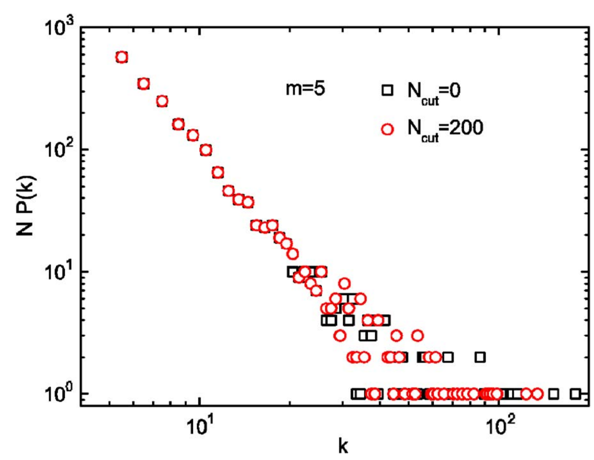

FIG. 2. (Color online) Degree distributions of networks before and after the decoupling process in the case of $m = 5$.

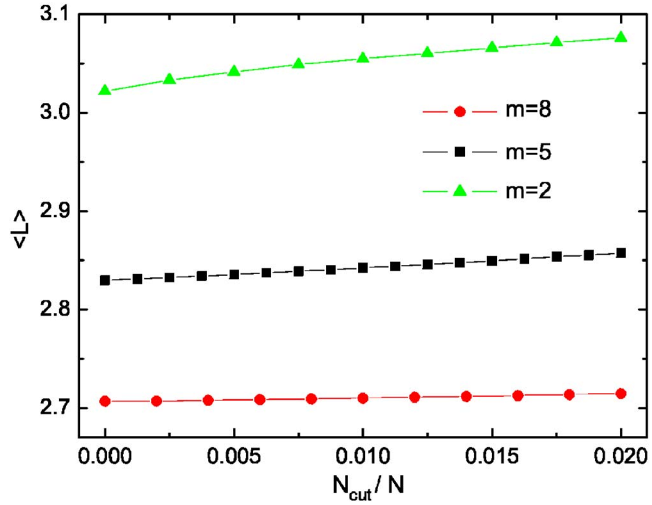

FIG. 3. (Color online) Average distance $\langle L\rangle$ as a function of N_{cut}/N for different values of m.

distance of a network has been deemed to be crucial for the synchronizability of the network, i.e., shorter average distance $\langle L\rangle$ results in stronger synchronizability for the same network size [8]. This is why SW networks are in favor of synchronization. We report $\langle L\rangle$ as a function of N_{cut}/N, as shown in Fig. 3. $\langle L\rangle$ is a slow increase function of N_{cut}/N, which demonstrates that the DP process has slight influences on $\langle L\rangle$ and $\langle L\rangle$ does not play the main role in the enhancement of the synchronizability [11].

High clustering coefficient (denoted by C) is a common property in many real networks [14]. Thus, we study the change of C induced by the DP. Figure 4 shows C'/C_0 depending on N_{cut}/N. The decreasing trend of C'/C_0 for different values of m is inconsistent with that of the synchronizability, which indicates C is not the most closely correlated with the synchronizability among the considered topological features. Moreover, C in original BA networks is very close to zero, the absolute change of C by the DP is neglectable.

Degree-degree (D-D) correlation is an important structural feature that classifies the real-world networks into two communities, i.e., social networks with positive correlation, and technological and biological networks with negative one [15]. A detailed definition can be seen in Ref. [15]. Figure 5 clearly shows that the synchronizability as a function of N_{cut}/N obviously depends on m, so the D-D correlation is not the best indicator for synchronizability.

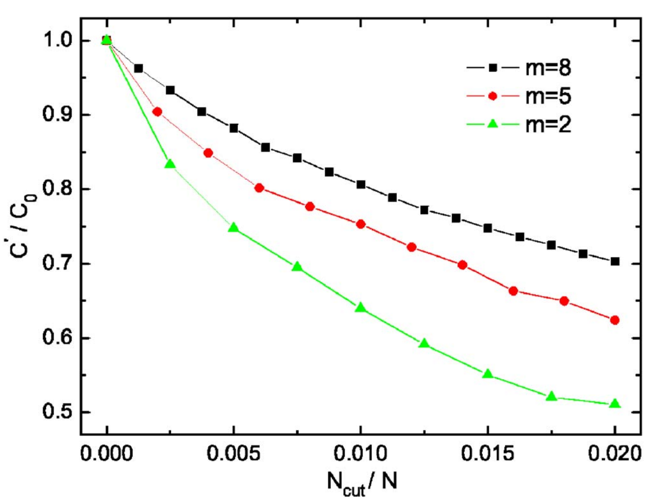

FIG. 4. (Color online) The change of average clustering coefficient C'/C_0 as a function of N_{cut}/N for different m.

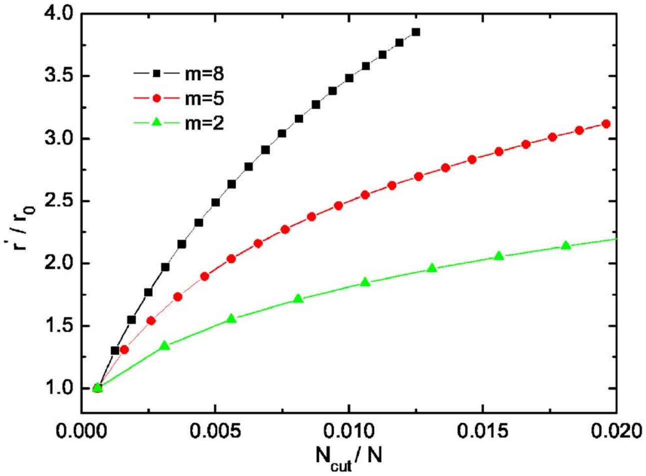

FIG. 5. (Color online) The change of degree-degree correlation r'/r_0 as a function of N_{cut}/N for different m.

Finally, we explore the maximum vertex betweenness $B_{\max}$ [14] depending on N_{cut}/N, as exhibited in Fig. 6. It is found that the decreasing trend of $B'_{\max}/B_0$ slightly depends on m, which is in accordance with that of synchronizability. Therefore, $B_{\max}$ seems to be a good factor to predict the synchronizability of BA networks. We further investigate the relationship between the change of synchronizability and that of $B_{\max}$. Simulation results are shown in the inset of Fig. 6. We find an universal correlation, i.e., $R'/R_0 \sim 0.65 B'_{\max}/B^0_{\max}$, which is independent of m. In Ref. [16], Pecora and Barahona have studied the synchronization behavior on several types of networks by adding edges to the

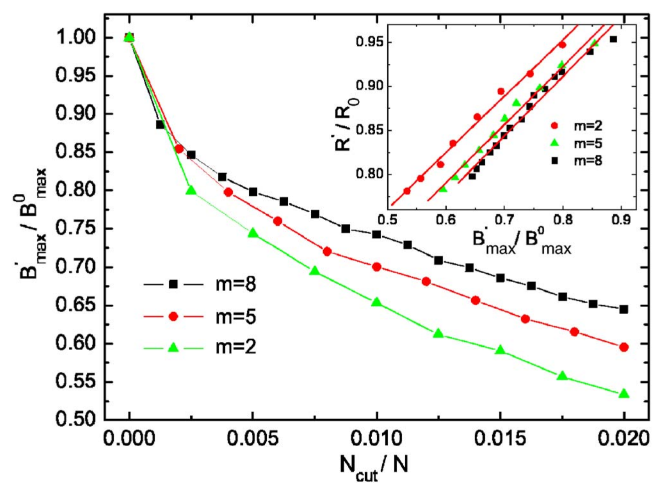

FIG. 6. (Color online) The change of maximum vertex betweenness $B'_{\max}/B^0_{\max}$ as a function of N_{cut}/N for different m. The inset shows the correlation between R'/R_0 and $B'_{\max}/B^0_{\max}$ for different m.

existent network. They found the ratio of maximum and minimum vertex degree helps determine the bound of eigenratio R. This finding can partially explain the effect of the DP on the synchronizability in the present work, where the DP usually performed among large degree vertices, so that the maximum degree is reduced while the minimum degree keeps unchanged, leading to better synchronizability. However, we have checked that removing edges between the maximum degree vertices and the minimum degree ones nearly has no influences on the network synchronizability. From a viewpoint of coupling information transmission, the edges among large degree vertices usually afford high traffic load, so that the coupling information transmission will be suppressed along these edges. In the language of traffic systems, congestion generally occurs at the edges among hubs. An effective way to alleviate the traffic congestion is to set up tollgates at both sides of the roads. This idea is essentially the same with the DP that allows the coupling information to avoid highly jammed edges to improve the transmission efficiency of the coupling information, so that the synchronizability of the network is enhanced. Perhaps vertex betweenness is the most reasonable way to measure traffic loads; that is why vertex betweenness seems to be the best predictor for the network synchronizability among the major structural parameters considered in this paper.

In conclusion, we have proposed a decoupling process performed in BA networks to enhance the network synchronizability. We have investigated the synchronizability in terms of the eigenratio of the coupling matrix affected by the decoupling process. Simulation results indicate that the decoupling process can effectively promote the synchronizability together with holding the scale-free structural property of the network. Some important statistical features of the network structure have been studied, including average distance, clustering coefficient, degree-degree correlation, as well as the maximum vertex betweenness. It is found that the maximum vertex betweenness seems to be the best indicator for the network synchronizability among the studied structural features by the comparison to the decreasing trend of the network synchronizability for different m. The effect of the decoupling process on the synchronization behavior is explained from the viewpoint of coupling information transmission. Our work also suggests that there are some essential relations between the network synchronization and the dynamics of traffic systems [17]; additionally, the decoupling process may have potential applications for its low cost in modifying the network structure.

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002).
[3] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[4] D. J. Watts, Annu. Rev. Sociol. **30**, 243 (2004).
[5] R. Pastor-Satorras and A. Vespignani, *Evolution and Structure of the Internet* (Cambridge University Press, Cambridge, England, 2004).
[6] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[7] D. J. Watts, *Small Worlds* (Princeton University Press, Princeton, NJ, 1999); L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A. Sigüenza, Phys. Rev. Lett. **84**, 2758 (2000); X. F. Wang, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 885 (2002); M. Timme, F. Wolf, and T. Geisel, Phys. Rev. Lett. **89**, 258701 (2002); J. Ito and K. Kaneko, Phys. Rev. E **67**, 046226 (2003); F. M. Atay, J. Jost, and A. Wende, Phys. Rev. Lett. **92**, 144101 (2004); Y. Moreno and A. F. Pacheco, Europhys. Lett. **68**, 603 (2004); P. G. Lind, J. A. C. Gallas, and H. J. Herrmann, Phys. Rev. E **70**, 056207 (2004).
[8] X. Guardiola, A. Diaz-Guilera, M. Llas, and C. J. Perez, Phys. Rev. E **62**, 5565 (2000).
[9] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
[10] A. E. Motter, C. Zhou, and J. Kurths, Phys. Rev. E **71**, 016116 (2005); M. Chavez, D.-U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett. **94**, 218701 (2005); C. Zhou, A. E. Motter, and J. Kurths, *ibid.* **96**, 034101 (2006).
[11] H. Hong, B. J. Kim, M. Y. Choi, and H. Park, Phys. Rev. E **69**, 067105 (2004); M. Zhao, T. Zhou, B.-H. Wang, and W.-X. Wang, *ibid.* **72**, 057102 (2005).
[12] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998); M. Barahona and L. M. Pecora, *ibid.* **89**, 054101 (2002).
[13] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[14] The detailed definition can be seen in Ref. [3].
[15] M. E. J. Newman, Phys. Rev. Lett. **89**, 208701 (2002).
[16] L. M. Pecora and M. Barahona, Chaos Complexity Lett., **1**(1), 61 (2005).
[17] W.-X. Wang, B.-H. Wang, C.-Y. Yin, Y.-B. Xie, and T. Zhou, Phys. Rev. E **73**, 026111 (2006).